

COMPLEX OF RELATIVELY HYPERBOLIC GROUPS

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ABSTRACT. In this article, we prove a combination theorem for a complex of relatively hyperbolic groups. It is a generalization of Martin's [5] work for combination of hyperbolic groups over a finite M_K -simplicial complex, where $k \leq 0$.

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1. INTRODUCTION

In [3], Dahamani showed that if G is fundamental group of an acylindrical finite graph of relatively hyperbolic groups with edge groups fully quasi-convex in the respective vertex groups, then G is hyperbolic relative to the images of the maximal parabolic subgroups of vertex groups and their conjugates in G . By gluing the relative hyperbolic boundaries of each local groups, Dahamani constructed a compact metrizable space ∂G on which G has convergence action and the limit points are either conical or bounded parabolic. So, G is a relatively hyperbolic group due to Bowditch [1]. Using these ideas, Martin [5] generalized this combination theorem for complex of hyperbolic groups. Let $G(\mathcal{Y})$ be a strictly developable non-positively curved simple complex of groups over a finite M_k simplicial complex with $k \leq 0$. Let G be the fundamental group of $G(\mathcal{Y})$ and X be a universal covering of $G(\mathcal{Y})$. Martin, in [5], proved that if X is hyperbolic, local groups are hyperbolic, local maps are quasiconvex embeddings and the action of G on X is acylindrical, then G is hyperbolic. In this article, we prove the relative hyperbolic version of Martin's result.

Theorem 1.1. *Let $\mathcal{G}(\mathcal{Y})$ be a strictly developable simple complex of groups over a finite M_κ -simplicial complex Y with $k \leq 0$ and satisfying the following properties:*

- *For each vertex v of Y , the vertex group G_v is relatively hyperbolic to the subgroup P_v .*
- *Local maps $\phi_{\sigma, \sigma'}$ are fully quasi-convex embeddings i.e. if $\sigma \subset \sigma'$ then $\phi_{\sigma, \sigma'}(G_{\sigma'})$ is fully quasiconvex in G_σ ,*
- *The universal covering X of $\mathcal{G}(\mathcal{Y})$ is hyperbolic.*
- *The action of G , the fundamental group of $\mathcal{G}(\mathcal{Y})$, on X is acylindrical.*

Then G is relatively hyperbolic to \mathcal{P} , where \mathcal{P} is the collection of the images of P_v in G under the natural embedding $G_v \hookrightarrow G$.

In section 2, we give two definitions of relatively hyperbolic groups due to Groves & Manning and Bowditch. Convergence action, fully quasiconvex subgroups, convergence property and finite intersection properties are given in this section. In section 3, acylindrical action of a group is defined and we give an example of acylindrically hyperbolic group due to Osin and Minasyan. Complex of groups is given in section 4 and in the subsequent section 5, the construction of boundary ∂G of fundamental group G of complex of groups $\mathcal{G}(\mathcal{Y})$ is provided. In section 6, we give the proof of Theorem 1.1.

2. RELATIVE HYPERBOLICITY AND CONVERGENCE ACTION

2.1. Relative Hyperbolicity. Relatively hyperbolic groups were first introduced by Gromov [7] to study hyperbolic manifolds with cusps. It was then studied by several people, we refer to the article [10] by Hruska for several equivalent notions of relatively hyperbolic groups. For our purpose, we will require two equivalent definitions of relative hyperbolicity due to Bowditch [1] and Groves & Manning [8].

Definition 2.1. (*Hyperbolic Metric Space*) Let $\delta \geq 0$. We say that a geodesic triangle Δ is δ -slim in a geodesic metric space if any side of the triangle Δ is contained in the δ -neighborhood of the union of the other two sides. A geodesic metric space is said to be δ -hyperbolic if all the triangles are δ -slim. A geodesic metric space is said to be hyperbolic if it is δ -hyperbolic for some $\delta \geq 0$.

First we give definition of relative hyperbolicity due to Groves & Manning. Let G be finitely generated group and \mathcal{P} be a finite collection of subgroups of it. Let \mathcal{S} be a finite generating set of G such that $\langle \mathcal{S} \cap P \rangle = P$ for all $P \in \mathcal{P}$ and Γ_G be the Cayley graph of G with respect to \mathcal{S} .

Definition 2.2. (*Combinatorial Horoballs*, [8]) Let C be a 1-complex with 0-skeleton C^0 and 1-skeleton C^1 . We will construct a 1-complex $\mathcal{H}(C)$ following ways:

- 0-skeleton of $\mathcal{H}(C)$, $\mathcal{H}(C)^{(0)} := C^{(0)} \times (\{0, 1, 2, \dots\})$,
- 1-skeleton of $\mathcal{H}(C)$, $\mathcal{H}(C)^{(1)} := \{[(v, 0), (w, 0)] : v, w \in C^{(0)}, [v, w] \in C^{(1)}\} \cup \{[(v, k), (w, k)] : v, w \in C^{(0)}, k > 0, d_C(v, w) \leq 2^k\} \cup \{[(v, k), (v, k+1)] : v \in C^{(0)}, k \geq 0\}$.

Definition 2.3. (*Augmented Space*, [10]) Let $G, \mathcal{P}, \mathcal{S}$ be as mentioned above. Also let \mathcal{T} be the set of representative for distinct cosets of all $P \in \mathcal{P}$. The Cayley graph of P with respect to $P \cap \mathcal{S}$ embedded in Γ_G as a subcomplex. Let Γ_t , $t \in \mathcal{T}$, be the translates of these subcomplexes. We define

$$\Gamma_G^h := \Gamma_G \cup \left(\bigcup_{t \in \mathcal{T}} (\mathcal{H}(\Gamma_t)) \right) / \simeq$$

as augmented space, where $\mathcal{H}(\Gamma_t) \times \{0\}$'s are identified to subcomplexes Γ_t .

Definition 2.4. (Groves & Manning, [8]) G is said to be hyperbolic relative to \mathcal{P} if the augmented space Γ_G^h is hyperbolic for any appropriate choice of \mathcal{S} .

Definition 2.5. (Convergence Group) Let G act on compact metrisable space M . The action is called convergence group action if for any sequence $\{g_n\}$ in G , there exists a subsequence $\{g_{\phi(n)}\}$ and $\xi^+, \xi^- \in M$ such that $g_{\phi(n)}(K)$ converges uniformly to ξ^+ , for all compact sets $K \subset M \setminus \{\xi^-\}$.

Definition 2.6. (1) (Bounded Parabolic Limit Points) An element $g \in G$ is called parabolic if it fixes exactly one point of M and the corresponding fixed point ξ (say) is said to be parabolic limit point. Furthermore, a parabolic limit point is said to be bounded parabolic if $\text{Stab}(\xi)$ acts properly discontinuously and cocompactly on $M \setminus \{\xi\}$.

(2) (Conical Limit Point) Let G have a convergence action on M . A point $\xi \in M$ is said to be conical limit point if there exists a sequence $\{g_n\}$ and $\xi^+, \xi^- \in M$ such that $g_n \xi \rightarrow \xi^+$, $g_n \xi' \rightarrow \xi^-$ for all $\xi' \in M \setminus \{\xi\}$.

(3) (Geometrically Finite Action) Let G have a convergence action on a compact metrisable space M . The action is said to be geometrically finite if the limit points are only conical and bounded parabolic.

Next we give the Bowditch's definition of Relative Hyperbolicity.

Definition 2.7. (Bowditch, [1]) Let G be a finitely generated group and \mathcal{P} be a finite collection of finitely generated subgroups of it. G is said to be hyperbolic relative to \mathcal{P} if it acts properly discontinuously on a proper hyperbolic metric space $\tilde{\Gamma}$ such that

- G acts on $\partial\tilde{\Gamma}$ by convergence and geometrically finite action.
- the conjugates of the elements of \mathcal{P} are precisely the maximal parabolic subgroups.

we call $\partial\tilde{\Gamma}$ the Bowditch boundary of G .

Remark 2.8. Due to equivalence of these definitions we can take Γ^h as $\tilde{\Gamma}$ and $\partial\Gamma^h$ will be Bowditch boundary.

2.2. Fully quasi-convex subgroup.

Definition 2.9. (Dahamani, [3]) Let G be a relatively hyperbolic group with Bowditch boundary ∂G . A subgroup H of G is called quasi-convex if H has a geometrically finite action on ΛH . It is called fully quasi-convex if for any infinite sequence $\{g_n\}$, all comes from distinct cosets of H , $\bigcap_n (g_n \Lambda H)$ is empty

Remark 2.10. If H is fully quasiconvex, then gHg^{-1} is fully quasi-convex $\forall g \in G$.

Remark 2.11. [3] Let G be a relatively hyperbolic group. If H is fully quasi-convex in G , then each parabolic point for H in $\Lambda(H)$ is a parabolic point for G in ∂G and

if P is the corresponding maximal parabolic subgroup in G then the corresponding maximal parabolic subgroup in H is precisely $P \cap H$.

The following two properties of fully quasi-convex subgroups are proved by F. Dahmani [3].

Theorem 2.12. (*Limit set property*) Let H_1 and H_2 are fully quasi-convex in G then $H_1 \cap H_2$ is fully quasi-convex. Moreover $\Lambda(H_1 \cap H_2) = \Lambda H_1 \cap \Lambda H_2$.

Theorem 2.13. (*Convergence property*) Let G be a relatively hyperbolic group and H be a fully quasi-convex subgroup in it. Let $\{g_n\}$ be a sequence of elements in G all comes from distinct cosets of H . Then there exists a subsequence $\{g_{\phi(n)}\}$ such that $g_{\phi(n)}\Lambda H$ uniformly converges to a point.

Next we will prove that there are finitely many conjugates of a fully quasi-convex subgroup which have infinite total intersection.

Proposition 2.14. (*Finite intersection property*) Let G be a relatively hyperbolic group and H be a fully quasi-convex subgroup in it. Then there exists finitely many distinct left cosets $g_1H, g_2H \dots g_mH$ in G for which $\bigcap_{k=1}^m g_kH g_k^{-1}$ is infinite.

Proof. If possible, let there exists a infinite sequence $\{g_n\}$ all comes from distinct cosets of H such that $\bigcap_{n=1}^{\infty} g_nH g_n^{-1}$ is infinite, i.e. $\Lambda(\bigcap_{n=1}^{\infty} g_nH g_n^{-1})$ is non empty. But $\Lambda(\bigcap_{n=1}^{\infty} g_nH g_n^{-1}) \subset \bigcap_{n=1}^{\infty} \Lambda(g_nH g_n^{-1})$ and the fact that $\Lambda(gHg^{-1}) = g\Lambda H$, we have $\bigcap_{n=1}^{\infty} g_n\Lambda H$ is non empty which contradicts the second condition of fully quasi-convexity. □

3. ACYLINDRICAL ACTION

An action of a group G on a simplicial tree T without inversions is said to be k -acylindrical if no non-trivial element of G fixes pointwise a segment of length k in T . A graph of group is called k -acylindrical if the action of the fundamental group on the Bass-Serre covering is k -acylindrical. More generally, a group G has an k -acylindrical action on a simplicial complex X if every subcomplex of X of diameter at least k has a finite pointwise stabiliser. In this section, we give theoretical examples of acylindrically hyperbolic groups, due to Osin and Minasyan, which is relevant to our relatively hyperbolic case.

A subgroup H of G is weakly normal, if there exists $g \in G$ for which $H \cap gHg^{-1}$ is finite. In [12], Osin and Minasyan proved the following theorem.

Theorem 3.1. [12] *Let $G = A *_C B$, $A \neq C \neq B$ and C is weakly malnormal in G . Then G is either virtually cyclic or acylindrically hyperbolic.*

Lemma 3.2. (Lemma 3.1 of [13]) *Given $\delta > 0$, there exist $D, C > 0$ such that the following holds:*

If x, y are points of a δ -hyperbolic metric space (X, d) , λ is a hyperbolic geodesic in X joining x, y , and π_λ is a nearest point projection of X onto λ with $d(\pi_\lambda(x), \pi_\lambda(y)) > D$, then $[x, \pi_\lambda(x)] \cup [\pi_\lambda(x), \pi_\lambda(y)] \cup [\pi_\lambda(y), y]$ lies in a C -neighborhood of any geodesic joining x, y .

The following lemma states that in a hyperbolic metric space if the distance between the nearest point projection of two points onto a quasiconvex set is sufficiently large then the geodesic segment joining two points come close to the quasiconvex set.

Lemma 3.3. *Given $\delta, k \geq 0$ there exist constants $D', C' > 0$ (depending on δ, k) such that the following holds: Let X be a δ -hyperbolic metric space and S be a Q -quasiconvex subset of X . For points $x, y \in X$, if $d(\pi_S(x), \pi_S(y)) > D'$ then there exist $p \in [x, y]$, $q \in S$ such that $d(p, q) \leq C'$.*

Proof. Let $D, C > 0$ be constants as in Lemma 3.2. Let $D' = D - 2(3\delta + Q)$ and λ be a geodesic segment joining $\pi_S(x)$ and $\pi_S(y)$.

First we prove that $d(\pi_S(x), \pi_\lambda(x))$ is bounded :

Consider the triangle $\triangle x\pi_S(x)\pi_\lambda(x)$. Since triangles are δ -thin, there exist $w_1 \in [x, \pi_S(x)]$, $w_2 \in [\pi_S(x), \pi_\lambda(x)]$, $w_3 \in [\pi_\lambda(x), x]$ such that $\text{diam}\{w_1, w_2, w_3\} \leq \delta$. As S is Q -quasiconvex, there exists w'_2 such that $d(w_2, w'_2) \leq Q$. Thus, as π_S is a nearest point projection, $d(w_1, \pi_S(x)) \leq \delta + Q$. Also $d(w_3, \pi_\lambda(x)) \leq \delta$. Therefore $d(\pi_S(x), \pi_\lambda(x)) \leq \delta + Q + d(w_1, w_3) + \delta \leq 3\delta + Q$.

Now if $d(\pi_S(x), \pi_S(y)) > D'$, then $d(\pi_\lambda(x), \pi_\lambda(y)) > D$. By Lemma 3.2, for any $r \in [\pi_\lambda(x), \pi_\lambda(y)]$, we have $d(r, [x, y]) \leq C$. Therefore there exists $q \in S$ such that $d(r, q) \leq Q$ and hence $B_{Q+C}(q)$ intersects $[x, y]$. Thus there exists $p \in [x, y]$ such that $d(p, q) \leq Q + C$. Taking $C' = Q + C$, we have the required result.

□

Proposition 3.4. *Let G be hyperbolic relative to a subgroup P and H be a relatively quasiconvex subgroup of infinite index in G . Then, H is weakly malnormal in G .*

Proof. Let X be the Cayley Graph of G with respect to a finite generating set. Consider the hyperbolic space X^h obtained by attaching the hyperbolic cones corresponding to each left coset of P in G . Let $H^b = X \setminus \bigcup \{H \cap gP : g \in G\}$ and $N(H)$ be the union of H^b and hyperbolic cones intersected by H . As H is relatively quasiconvex, $N(H)$ is k -quasiconvex set in X^h for some $k \geq 0$ (see Lemma 1.21 of

[16]). Let $g \in G$ be such that $d_{X^h}(N(H), gN(H)) > C + k$ for all $x \in X^h$. We prove that for this $g \in G$, $H \cap gHg^{-1}$ is finite.

By Lemma 3.3 there exists $D > 0$ such that for any two $x, y \in gN(H)$, $d_{X^h}(\pi_{N(H)}(x), \pi_{N(H)}(y)) \leq D$. If $H \cap gHg^{-1}$ contains a parabolic element, then there exists $x \in G$ such that $H \cap gHg^{-1} \cap xPx^{-1}$ is non-empty and contains a parabolic element. This implies that $N(H)$ and $gN(H)$ share the common hyperbolic cone corresponding to xPx^{-1} and so the distance $d_{X^h}(N(H), gN(H)) = 0$, which contradicts the choice of g .

Now, if $H \cap gHg^{-1}$ is hyperbolic then there exists hyperbolic elements $h_1, h_2 \in H$ such $h_1 = gh_2g^{-1}$ and $d_{X^h}(z, h_2z) > D$ for all $z \in X^h$. Let $x \in N(H)$, there exists $y \in N(H)$ such that $d_{X^h}(x, gN(H)) = d_{X^h}(x, gy)$. As G acts by isometry on X^h , $d_{X^h}(x, gy) = d_{X^h}(h_1x, h_1gy)$ and these distances are greater than $C + k$. H stabilises $N(H)$, so $h_1x \in N(H)$ and $h_1gy = gh_2y \in gN(H)$. By Lemma 3.3 we have $d_{X^h}(gy, gh_2y) \leq D$. So, $d_{X^h}(y, h_2y) \leq D$ which contradicts the choice of h_2 . \square

As an application of it, we have the following corollary

Corollary 3.5. *Let $G = A *_C B$ and C_1 be a subgroup of C . Suppose A is hyperbolic relative to the subgroup C_1 and C is relatively quasiconvex of infinite index in A and B is arbitrary. Then G is either virtually cyclic or acylindrically hyperbolic.*

Proof. As C is relatively quasiconvex therefore it is weakly malnormal. Then the result follows from the Theorem 3.1 \square

4. COMPLEX OF GROUPS AND COMPLEX OF SPACES

In this section we will discuss the basics of complex of groups which is a generalization of graph of groups. For a detailed discussion on this topic we refer to [2].

Let Y be a simplicial complex. We will denote the set of simplices and set of vertices of Y by $S(Y)$ and $V(Y)$ respectively. Let \mathcal{Y} be the scwol (refer to [2]) corresponding to the first Barrycentric subdivision of Y and its directed edge set is denoted by $\mathcal{E}^\pm(\mathcal{Y})$.

4.1. Complex of Groups.

Definition 4.1. *A simple complex of groups, $G(\mathcal{Y})$, over a simplicial complex Y consists of*

- (1) *local groups G_σ for each $\sigma \in S(Y)$*
- (2) *a monomorphism $\varphi_{\sigma, \sigma'} : G_{\sigma'} \rightarrow G_\sigma$ whenever $\sigma \subset \sigma'$.*
- (3) *for $\sigma \subset \sigma' \subset \sigma''$, $\varphi_{\sigma, \sigma''} = \varphi_{\sigma, \sigma'} \circ \varphi_{\sigma', \sigma''}$*

Definition 4.2. (*Fundamental Group of Complex of Groups*) Let T be a maximal tree in 1-skeleton of \mathcal{Y} . The Fundamental Group of $G(\mathcal{Y})$ with respect to T , denoted by $\pi_1(G(\mathcal{Y}), T)$, is generated by $\bigsqcup_{\sigma \in S(Y)} G_\sigma \bigsqcup \mathcal{E}^\pm(\mathcal{Y})$ subject to

- (1) relations of G_σ ,
- (2) $(a^+)^{-1} = a^-, (a^-)^{-1} = a^+$,
- (3) $(ab)^+ = a^+b^+$
- (4) $a^+ga^- = \varphi_a(g)$,
- (5) $a^+ = 1$ for all edge a of T .

In fact the above definition is independent of the choice of the maximal tree and we will call it G in the subsequent sections. There is a canonical morphism, $\iota_T : G(\mathcal{Y}) \rightarrow G$ which takes $G_\sigma \rightarrow G$ injectively and $a \mapsto a^+$.

Next we will define a CW complex on which G will act naturally and the quotient space will be Y .

Definition 4.3. (*Universal Covering*) We define the universal covering of $G(\mathcal{Y})$ associated to ι_T as

$$X := \left(G \times \coprod \sigma \right) / \simeq$$

where $(g, i_{\sigma, \sigma'}(x)) \simeq (g\iota_T([\sigma, \sigma'])^{-1}, x)$, $[\sigma, \sigma'] \in \mathcal{E}(\mathcal{Y})$, $i_{\sigma, \sigma'} : \sigma' \rightarrow \sigma$ is the embedding and $(gg', x) \simeq (g, x)$, $g' \in G_\sigma, g \in G$.

G acts naturally on X by left multiplication on the first factor.

4.2. Complex of Spaces.

Definition 4.4. A complex of spaces, $C(\mathcal{Y})$, over a simplicial complex Y consists of

- (1) local spaces C_σ for each $\sigma \in S(Y)$
- (2) an embedding $\varphi_{\sigma, \sigma'} : C_{\sigma'} \rightarrow C_\sigma$ whenever $\sigma \subset \sigma'$.
- (3) for $\sigma \subset \sigma' \subset \sigma''$, $\varphi_{\sigma, \sigma''} = \varphi_{\sigma, \sigma'} \circ \varphi_{\sigma', \sigma''}$

Definition 4.5. (*Realisation of complex of spaces*) Let $C(\mathcal{Y})$ be a complex of spaces over X . We define the realisation of $C(\mathcal{Y})$ to be the quotient space

$$|C(\mathcal{Y})| := \left(\coprod \sigma \times C_\sigma \right) / \simeq$$

where $(i_{\sigma, \sigma'}(x), s) \simeq (x, \varphi_{\sigma, \sigma'}(s))$, $[\sigma, \sigma'] \in \mathcal{E}(\mathcal{Y})$

5. CONSTRUCTION OF EG AND ∂G

Let $G(\mathcal{Y})$ be a developable simple complex of group with fundamental group G as defined in 4.2. For each vertex v of Y , the vertex group G_v is relatively hyperbolic to the subgroup P_v . Local maps $\phi_{\sigma, \sigma'}$ are fully quasi-convex embeddings i.e. if $\sigma \subset \sigma'$

then $\phi_{\sigma,\sigma'}(G_{\sigma'})$ is fully quasiconvex in G_σ . Then by Remark 2.11 G_σ is relatively hyperbolic to the subgroup $P_v \cap G_\sigma$ for each $\sigma \in S(X)$. We call $P_v \cap G_\sigma$ as P_σ . By extending the generating set of $G_{\sigma'}$ to a generating set of G_σ $\phi_{\sigma,\sigma'} : G_{\sigma'} \rightarrow G_\sigma$ will induce a natural equivariant embeddings between the corresponding Cayley graphs and Augmented spaces.

Let X be the universal covering of $G(\mathcal{Y})$ associated to ι_T . Let Γ_σ be the Cayley graph of G_σ and Γ_σ^h be the proper hyperbolic spaces on which G_σ acts properly discontinuously according to Bowditch definition. Also let ∂G_σ be the Bowditch Boundary of G_σ .

Definition 5.1. *we define a complex of spaces over X , EG (resp. EG^h) associated to $G(\mathcal{Y})$*

$$EG := \left(G \times \coprod (\sigma \times \Gamma_\sigma) \right) / \simeq$$

$$EG^h := \left(G \times \coprod (\sigma \times \Gamma_\sigma^h) \right) / \simeq$$

where $(g, i_{\sigma,\sigma'}(x), s) \simeq (g\iota_T([\sigma, \sigma']^{-1}, x, \varphi_{\sigma,\sigma'}(s)), [\sigma, \sigma'] \in \mathcal{E}(\mathcal{Y})$ and $(gg', x, s) \simeq (g, x, g's), g' \in G_\sigma, g \in G$.

G has natural action on EG^h by left multiplication on the first factor. Also there is a obvious projection map $p : EG^h \rightarrow X$ which injectively sends the first two factors and this map is G -equivariant.

Definition 5.2. *We define the space*

$$\partial_{stab}G := \left(G \times \coprod (\{\sigma\} \times \partial G_\sigma) \right) / \simeq$$

where $(g, \{\sigma\}, s) \simeq (g\iota_T([\sigma, \sigma']^{-1}, \{\sigma'\}, \varphi_{\sigma,\sigma'}(s)), [\sigma, \sigma'] \in \mathcal{E}(\mathcal{Y})$ and $(gg', \{\sigma\}, s) \simeq (g, \{\sigma\}, g's), g' \in G_\sigma, g \in G$.

Now we define the boundary of G as

$$\partial G := \partial_{stab}G \cup \partial X$$

Also we define $\overline{EG^h} := EG^h \cup \partial G$.

Here, we are taking the union of augmented spaces (resp. boundaries) corresponding to vertex groups of X and glueing them along the augmented spaces (resp. boundaries) of the local groups accordingly.

G also has natural action on ∂G and $\overline{EG^h}$ by left multiplication on the first factor. In the subsequent section we will try to give a topology on $\overline{EG^h}$ such that $\overline{EG^h}$ and ∂G will be compact and action of G will be geometrically finite convergence action.

For simplicity of notation we will denote G_σ as the stabilizer subgroup of the simplex σ in X (Note that $stab(\sigma)$ is actually conjugate of a local group of $G(\mathcal{Y})$).

In the subsequent sections we assume our complex of groups satisfies all the hypothesis of the main theorem. Then by 2.12, 2.13 and 2.14 $\mathcal{G}(\mathcal{Y})$ will satisfies *limit set property*, *convergence property* and *finite intersection property*.

5.1. Domains and Topology.

Definition 5.3. Let $\xi \in \partial_{stab}G$. We define domain of ξ , $D(\xi) := \text{span}\{\sigma \in S(X) | \xi \in \partial_{stab}G_\sigma\}$.

Proposition 5.4. (Martin,[5]) For every $\xi \in \partial_{stab}G$, $D(\xi)$ is finite convex subcomplex of X uniformly bounded by the acylindricity constant.

Existence of such a neighbourhood proved by A. Martin[5].

Definition 5.5. (ξ -family) Let $\xi \in \partial_{stab}G$. A ξ -family is defined to be as a collection \mathcal{U} of open sets U_v where $v \in \text{Vert}(D(\xi))$ and U_v is a neighborhood of representative of ξ in $\overline{\Gamma_v^h}$ such that for every two adjacent vertices v, v' we have

$$\varphi_{v,e}(\overline{\Gamma_e^h}) \cap U_v = \varphi_{v',e}(\overline{\Gamma_e^h}) \cap U_{v'}, \text{ where } e \text{ is an edge between } v \text{ and } v'$$

Next, we give a topology on ∂G due to Martin [5].

Let us choose a basepoint $v_0 \in X$. For a given point $x \in X$ (resp. $\eta \in \partial X$) we denote c_x (resp. c_η) to be the unique geodesic segment (resp. geodesic ray) from v_0 to x (resp. η). We denote $D^\epsilon(\xi)$ to be the ϵ -neighbourhood of $D(\xi)$ where $\epsilon \in (0, 1)$.

A geodesic c is said to be *goes through* (reps. *enters*) $D^\epsilon(\xi)$ if $\exists t_0, t_1$ such that $c(t_0) \in D^\epsilon(\xi), c(t_1) \in \overline{D^\epsilon(\xi)}$ and $\forall t > t_1, c(t) \notin D^\epsilon(\xi)$ (reps. if $\exists t_0$ such that $c(t_0) \in D^\epsilon(\xi)$). If c_x or c_η goes through $D^\epsilon(\xi)$, the h is met by first simplex which is met by c_x or c_η after leaving $D^\epsilon(\xi)$ is said to be an *exit simplex* and is denoted by $\sigma_{\xi,\epsilon}(x)$. For $x \in D^\epsilon(\xi)$ we define $\sigma_{\xi,\epsilon}(x) := \sigma_x$

Definition 5.6. (Martin [5]) Let $\xi \in \partial_{stab}G, \mathcal{U}$ a ξ -family and $\epsilon \in (0, 1)$. We define

- (i) $\text{Cone}_{\mathcal{U},\epsilon}(\xi) := \{x \in \overline{X} \setminus D^\epsilon(\xi) | c_x \text{ goes through } D^\epsilon(\xi) \text{ and for all } v \in V(D(\xi) \cap \sigma_{\xi,\epsilon}(x)), \overline{\Gamma_{\sigma_{\xi,\epsilon}(x)}^h} \subset U_v, \text{ in } \overline{\Gamma_v^h}\}$,
- (ii) $\widetilde{\text{Cone}}_{\mathcal{U},\epsilon}(\xi) := \{x \in \overline{X} | c_x \text{ enters } D^\epsilon(\xi) \text{ and for all } v \in V(D(\xi) \cap \sigma_{\xi,\epsilon}(x)), \overline{\Gamma_{\sigma_{\xi,\epsilon}(x)}^h} \subset U_v, \text{ in } \overline{\Gamma_v^h}\}$

Martin, in [5], proved that the cones $\text{Cone}_{\mathcal{U},\epsilon}(\xi)$ and $\widetilde{\text{Cone}}_{\mathcal{U},\epsilon}(\xi)$ are open sets in \overline{X}

Topology on $\overline{EG^h}$.

$\overline{EG^h}$ consists of three kind of elements $\tilde{x} \in EG^h, \eta \in \partial X$ and $\xi \in \partial_{stab}G$.

- For $\tilde{x} \in EG^h$: We define a basis of neighbourhood of \tilde{x} in $\overline{EG^h}$ coming from the topology of EG^h as a CW complex and denote it by $\mathcal{O}_{\overline{EG^h}}(\tilde{x})$.
- For $\eta \in \partial X$: Let $\mathcal{O}_{\overline{X}}(\eta)$ be the basis of neighbourhood of η in \overline{X} and $U \in \mathcal{O}_{\overline{X}}(\eta)$. we define a neighbourhood of η in $\overline{EG^h}$

$$V_U(\eta) = p^{-1}(U \cap X) \cup (U \cap \partial X) \cup \{\xi \in \partial_{stab} G \mid D(\xi) \subset U\}$$

We define, $\mathcal{O}_{\overline{EG^h}}(\eta) := \{V_U(\eta) \mid U \in \mathcal{O}_{\overline{X}}(\eta)\}$, the basis of neighbourhood of η in $\overline{EG^h}$.

- For $\xi \in \partial_{stab} G$: Let \mathcal{U} be ξ -family and $\epsilon \in (0, 1)$. We define three sets around ξ as follows:
 $W_1 = \{\tilde{x} \in EG^h : p(\tilde{x}) = x \in D^\epsilon(\xi) \text{ and } \varphi_{v, \sigma_x}(\tilde{x}) \in U_v \text{ for all vertex } v \in D(\xi) \cap \sigma_x\},$
 $W_2 := Cone_{\mathcal{U}, \epsilon}(\xi) \cap \overline{X},$
 $W_3 := \{\xi' \in \partial_{stab} G : D(\xi') \setminus D(\xi) \subset \widetilde{Cone_{\mathcal{U}, \epsilon}(\xi)} \text{ and } \xi' \in U_v \text{ for all vertex } v \in D(\xi) \cap D(\xi')\}$

We define a neighbourhood around ξ as $W_{\mathcal{U}, \epsilon}(\xi) := W_1 \cup W_2 \cup W_3$. Let $\mathcal{O}_{\overline{EG^h}}(\xi) = \{W_{\mathcal{U}, \epsilon}(\xi) : \mathcal{U} \text{ is } \xi\text{-family and } \epsilon \in (0, 1)\}$. We give $\overline{EG^h}$ the topology generated by the sub-basis $\mathcal{O}_{\overline{EG^h}}(x)$, $x \in \overline{EG^h}$. In fact Martin showed that $\mathcal{O}_{\overline{EG^h}}(x)$ is a basis for this topology. Note that the definition of neighbourhoods depends on the basepoint chosen but Martin in [5] showed that the topology remains equivalent by changing the basepoint. Martin [5] showed that $\overline{EG^h}$ is separable, metrisable and is compact.

6. MAIN THEOREM

Let $\mathcal{G}(\mathcal{Y})$ be a strictly developable simple complex of groups over a finite M_κ -simplicial complex Y with $k \leq 0$ and satisfying the hypothesis of Theorem 1.1. Let G be the fundamental group of $\mathcal{G}(\mathcal{Y})$.

Local groups G_v are relatively hyperbolic implies G_v has convergence action on the relative hyperbolic boundary ∂G_v and ∂G is sequentially compact.

Lemma 6.1. (Lemmas 9.14, 9.15, 9.16 of [5]) G has convergence action on ∂G

Lemma 6.2. (Lemma 9.18 of [5]) The conical limit points of G are precisely the conical limit points vertex stabilizers and boundary points of X

Lemma 6.3. (i) The image of a bounded parabolic point in vertex stabilizer's boundary is a bounded parabolic for G ,

(ii) The corresponding maximal parabolic subgroup is the image of a maximal parabolic subgroup in the vertex stabilizer.

Proof. (i) Let $\tilde{\xi}$ be a bounded parabolic point of boundary of some vertex stabilizer and $\pi(\tilde{\xi}) = \xi$ be its image in ∂G . We will show ξ is bounded parabolic.

Let $P = stab(\xi)$ in G . Then P stabilizes $D(\xi)$, domain of ξ . Let $\xi_{v_i} \in \partial G_{v_i}$ be such that $\pi(\xi_{v_i}) = \xi$, where $\{v_1, \dots, v_n\}$ is the set of vertices of $D(\xi)$. From construction of $\partial_{stab} G$, for each $i = 1, \dots, n$, ξ_{v_i} is bounded parabolic point of G_{v_i} and let P_{v_i} be the maximal parabolic subgroup of G_{v_i} stabilising ξ_{v_i} . From the construction of $D(\xi)$, $\xi_{v_1}, \dots, \xi_{v_n}$ are the all which are identified to ξ . Thus, P_{v_i} also stabilizes $\{\xi_{v_1}, \dots, \xi_{v_n}\}$

and hence it stabilizes $D(\xi)$. So, P_{v_i} is a subgroup of P . Let K_i be a compact fundamental domain in $\partial G_{v_i} \setminus \{\xi_{v_i}\}$ for cocompact action of P_{v_i} on $\partial G_{v_i} \setminus \{\xi_{v_i}\}$. Let $N(D(\xi))$ be one open simplicial neighborhood of $D(\xi)$ in X and $S(N(D(\xi)) \setminus D(\xi))$ be collection of simplices in $N(D(\xi))$ of $D(\xi)$ that is not contained in $D(\xi)$. Let $S_i := \{\sigma \in S(N(D(\xi)) \setminus D(\xi)) : \partial G_\sigma \cap K_i \neq \emptyset\}$.

We claim that $\bigcup_{i=1}^n PS_i = S(N(D(\xi)) \setminus D(\xi))$. Let $\sigma \in S_i$ and $p \in P$. As P stabilizes $D(\xi)$, then $p\sigma \in S(N(D(\xi)) \setminus D(\xi))$. Conversely, let $\sigma \in (N(D(\xi)) \setminus D(\xi))$ and $v_i \in D(\xi) \cap \sigma$. Then $\partial G_\sigma \subset G_{v_i} \setminus \{\xi_{v_i}\}$. But, K_i is a fundamental domain for P_{v_i} hence there exists $p \in P_{v_i} \hookrightarrow P$ such that $p\partial G_\sigma = \partial G_{p\sigma}$ intersect with K_i . So, $p\sigma \in S_i$ and this proves our claim.

$D(\xi)$ is a finite convex subspace of the $CAT(0)$ space X and is stabilized by P . Hence, P has a fix point, say $\{x_0\}$, in $D(\xi)$. The topology on ∂G is independent of base point. Let us take x_0 to be the basepoint for the topology of ∂G . For $x \in X \setminus D(\xi)$, let $\sigma_x \in (N(D(\xi)) \setminus D(\xi))$ denote the exit simplex for the geodesic $[x_0, x]$.

- For each i , let $T_i := \{x \in X : \sigma_x \in S_i\}$ and let $\overline{T_i}$ be its closure in \overline{X} .
 - Let $K'_i := \{\alpha \in \partial G : D(\alpha) \cap \overline{T_i} \neq \emptyset\}$ and $\overline{K'_i}$ be its closure in ∂G .
- For each i , $K_i \cup \overline{K'_i}$ being closed is compact in ∂G . We claim $\xi \notin (K_i \cup \overline{K'_i})$ for all i and $\bigcup_{i=1}^n (K_i \cup \overline{K'_i})$ is a compact fundamental domain for action of P on $\partial G \setminus \{\xi\}$.

Claim 1. $\xi \notin (K_i \cup \overline{K'_i})$.

Obviously $\xi \notin K_i, K'_i$ for any i . Now if possible let $\{\alpha_m\}$ be a sequence in K'_i for some i such that $\alpha_m \rightarrow \xi$. By the definition of the topology on ∂G , $D(\alpha_m) \setminus D(\xi) \subset Cone_{\mathcal{U}, \epsilon}(\xi)$ for any ξ -family \mathcal{U} and $\epsilon > 0$. Let $x_m \in D(\alpha_m) \setminus D(\xi)$ then by definition of K' , $\partial G_{\sigma_{x_m}} \cap K_i \neq \emptyset$ for all m . Also, $\partial G_{\sigma_{x_m}} \subset G_{v_i} \setminus \{\xi_{v_i}\}$, by convergence Property $\partial G_{\sigma_{x_m}} \rightarrow \xi_{v_i}$ uniformly. This implies $\xi_{v_i} \in K_i$, which is a contradiction.

Claim 2. $\bigcup_{i=1}^n P(K_i \cup \overline{K'_i}) = \partial G \setminus \{\xi\}$.

Let $\alpha (\neq \xi) \in \partial G$. If $\alpha \in \partial X$ then the claim is true since x_0 is fixed by P and $\bigcup_{i=1}^n PS_i = S(N(D(\xi)) \setminus D(\xi))$. For $\alpha \in \partial_{stab} G$ we will divide the proof of the claim into two cases.

Case 1. $D(\alpha) \cap D(\xi) \neq \emptyset$. Then $\alpha \in \partial G_{v_i}$ for some $v_i \in D(\alpha) \cap D(\xi)$. $\alpha \neq \xi_{v_i}$, now since K_i is a fundamental domain for the action of P_{v_i} in $\partial G_{v_i} \setminus \{\xi_{v_i}\}$, there exists $x \in K_i$ and $p \in P_{v_i} \hookrightarrow P$ such that $\alpha = px \in PK_i$.

Case 2. $D(\alpha) \cap D(\xi) = \emptyset$. Let $x \in D(\alpha)$ and $\sigma_x \in N(D(\xi)) \setminus D(\xi)$ be the exit simplex for the geodesic $[x_0, x]$ in X . As $\bigcup_{i=1}^n PS_i = S(N(D(\xi)) \setminus D(\xi))$ and P fixes x_0 there exists $p \in P$ such that $\sigma_{px} = p\sigma_x \in S_i$ for some i . So, $px \in T_i$ and $px \in pD(\alpha) = D(p\alpha)$. So, $p\alpha \in K'_i$ and hence $\alpha \in PK'_i$.

(ii) Let $\tilde{\xi}$ be a bounded parabolic point of boundary of some vertex stabilizer and $\pi(\tilde{\xi}) = \xi$ be its image in ∂G , with $P = \text{stab}(\xi)$ in G . Then P stabilizes $D(\xi)$ and it fixes a point $x_0 \in D(\xi)$. Let σ be the simplex in $D(\xi)$ containing x_0 in the interior. From the definition of action of G on X , if some element of G fixes an interior point of a simplex then it fixes the whole simplex pointwise. So, P fixes σ pointwise. Without loss of generality, we can take x_0 to be a vertex v_i of σ . Thus P fixes ξ_{v_i} and hence $P = P_{v_i}$.

□

Proof of Theorem 1.1: From Lemma 6.1, G has a convergence action on compact metrizable space ∂G . The limit points are either conical (by Lemma 6.2) or bounded parabolic (by Lemma 6.3). Hence, by definition 2.7, G is hyperbolic relative to \mathcal{P} , where \mathcal{P} is the collection of the images of P_v in G under the natural embedding $G_v \hookrightarrow G$.

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